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Transitive A_6-invariant k-arcs in $PG(2, q)$

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Abstract

For $q = p^r$ with a prime $p \geq 7$ such that $q \equiv 1$ or 19 (mod 30), the desarguesian projective plane $PG(2, q)$ of order q has a unique conjugacy class of projectivity groups isomorphic to the alternating group A_6 of degree 6. For a projectivity group $\Gamma \cong A_6$ of $PG(2, q)$, we investigate the geometric properties of the (unique) Γ-orbit $\mathcal{O}$ of size 90 such that the 1-point stabilizer of Γ in $\mathcal{O}$ is a cyclic group of order 4. Here $\mathcal{O}$ lies either in $PG(2, q)$ or in $PG(2, q^2)$ according as 3 is a square or a non-square element in $GF(q)$. We show that if $q \geq 349$ and $q \neq 421$, then $\mathcal{O}$ is a 90-arc, which turns out to be complete for $q = 349, 409, 529, 601, 661$. Interestingly, $\mathcal{O}$ is the smallest known complete arc in $PG(2, 601)$ and in $PG(2, 661)$. Computations are carried out by MAGMA.

Keywords: finite desarguesian planes, k-arcs, $PSL(2, 9)$.

1 Introduction

Let $GF(q)$ be a finite field of order $q = p^r$, a power of an odd prime p. In the projective plane $PG(2, q)$ coordinatized by $GF(q)$, a $k-$*arc* K is a set of k points no three of which are collinear. If an arc of $PG(2, q)$ is not contained in a larger arc in $PG(2, q)$ then it is called *complete*. From the theory of linear codes, every $k-$arc of $PG(2, q)$ corresponds to a $[k, 3, k-2]$ *maximum distance separable* (MDS) code of length k, dimension 3 and minimum distance $k - 2$. This gives a motivation for the the study of k-arcs in $PG(2, q)$; those with many projectivities were investigated in several papers, see [4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 17, 18, 19].

The maximum size of a (complete) arc in $PG(2, q)$ is $q+1$, and the points of an irreducible conic in $PG(2, q)$ form an arc of size $q+1$. Actually, such $(q+1)$-arcs arising from irreducible conics are the unique $(q+1)$-arcs in $PG(2, q)$. This is the famous Segre's theorem [20]; see also [10, Theorem 8.7]. Therefore, the projectivity group which preserves a $(q+1)$-arc K in $PG(2, q)$ is isomorphic to the projective linear group $PGL(2, q)$ and acts on K as $PGL(2, q)$ in its natural 3-transitive permutation representation. In particular, every $(q + 1)$-arc K is transitive. Here, the term of a *transitive* arc of $PG(2, q)$ is adopted to denote a k-arc K such that the projectivity group preserving K acts transitively on the points of K.

Let Γ be a finite group which can act faithfully as a projectivity group in $PG(2,q)$. Actually, this may happen in different characteristics p. For instance, $PG(2,q)$ with $p \neq 5$ has a projectivity group isomorphic to the alternating group A_6 if and only if $q \equiv 1$ or $19 \pmod{30}$, and in this case such a projectivity group is uniquely determined up to conjugacy in $PGL(3,q)$, see [2]. So the question arises whether or not a Γ-invariant arc of a fixed size k exists in $PG(2,q)$ for infinitely many values of p. From previous work, the answer is affirmative for $\Gamma \cong A_6$ and $k = 72$, see [14], and $\Gamma \cong PSL(2,7)$ and $k = 42$ see [16]. However the answer is negative for the Hesse-group of order 216 for any $k \geq 9$, see [21].

In this paper we investigate the case of $\Gamma \cong A_6$ and $k = 90$, giving a positive answer to the above question:

Theorem 1.1. *For a power q of a prime $p \geq 7$ such that $q \equiv 1$ or $19 \pmod{30}$, let $\Gamma \cong A_6$ be a projectivity group of $PG(2,q)$. Let $\mathcal{O}$ be the (unique) Γ-orbit of length 90 in $PG(2,q)$ such that the 1-point stabilizer of Γ in $\mathcal{O}$ is a cyclic group of order 4. Then $\mathcal{O}$ is a 90-arc in $PG(2,q)$ except for a few cases where*

(i) $q = 61$ *and $\mathcal{O}$ is a set of type* $(0,1,2,4,6)$;

(ii) $q = 109$ *and $\mathcal{O}$ is a set of type* $(0,1,2,3)$;

(iii) $q = 181$ *and $\mathcal{O}$ is a set of type* $(0,1,2,3)$;

(iv) $q = 229$ *and $\mathcal{O}$ is a set of type* $(0,1,2,4)$;

(v) $q = 241$ *and $\mathcal{O}$ is a set of type* $(0,1,2,4)$;

(vi) $q = 421$ *and $\mathcal{O}$ is a2 set of type* $(0,1,2,3)$;

(vii) $q = 7^2$ *and $\mathcal{O}$ is a set of type* $(0,1,2,4)$;

(viii) $q = 11^2$ *and $\mathcal{O}$ is a set of type* $(0,1,2,5)$;

(ix) $q = 13^2$ *and $\mathcal{O}$ is a set of type* $(0,1,2,4)$;

(x) $q = 17^2$ *and $\mathcal{O}$ is a set of type* $(0,1,2,3)$;

(xi) $q = 19^2$ *and $\mathcal{O}$ is a set of type* $(0,1,2,5)$;

An exhaustive computer aided search shows that such a 90-arc may be complete for some particular values of q, namely $q = 349, 409, 529, 601, 661$. It is worth mentioning that this gives the smallest known complete arc in $PG(2,601)$ and in $PG(2,661)$, see [1, 7].

Notation and terminology are standard, see [10]. Furthermore, q always denotes a power of an odd prime $p \geq 7$ such that $q \equiv 1$ or $19 \pmod{30}$. Then 3 divides $q-1$ and 5 is a square element in the multiplicative group of $GF(q)$. The latter two requirement are indeed necessary and sufficient for $PGL(3,q)$ to have a subgroup $\Gamma \cong A_6$.

2 Preliminary Results

We give an explicit representation of Γ as a subgroup of $PGL(3,q)$ using the well known isomorphism $A_6 \cong PSL(2,9)$. Following [14], we choose a primitive

element η in $GF(9)$ satisfying $\eta^2 = \eta + 1$, and introduce the following matrices over $GF(9)$,

$$\mathsf{U}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathsf{U}_2 = \begin{pmatrix} 1 & \eta^2 \\ 0 & 1 \end{pmatrix}, \mathsf{V} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathsf{W} = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-}1 \end{pmatrix}.$$

It is easy to show that the above matrices generate $SL(2, 9)$. Furthermore, V^4 is the identity matrix I.

The factor group $SL(2, 9)/\langle -\mathsf{I} \rangle$ is $PSL(2, 9)$.

Let $\Phi : SL(2, 9) \to PSL(2, 9)$ be the associated natural homomorphism, and set $M = \Phi(\mathsf{M})$ with $\mathsf{M} \in SL(2, 9)$.
There is a unique conjugacy class of elements of order 4 in $PSL(2, 9)$, and the projectivity W with matrix representation W is such an element of order 4 (then $\langle W \rangle$ has order 4 ...si potrebbe aggiungere).

Now, fix a primitive third root t of unity in $GF(q)$ and an element z such that $z^2 = 5$. Let $\Delta = t - t^2$. Define the following matrices over $GF(q)$:

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} -2 & 1 + \Delta z & 1 + \Delta z \\ 1 - \Delta z & 4 & -2 \\ 1 - \Delta z & -2 & 4 \end{pmatrix}, \ \mathbf{W} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & t^2 & t \end{pmatrix}.$$

Let $\bar{U}, \bar{\Omega}, \bar{V}$ and $\bar{W}$ be the associated projectivities of $PGL(3, q)$. From [14, Theorem 2.6], the projectivity group generated by $\bar{U}, \bar{\Omega}, \bar{V}$ and $\bar{W}$ is isomorphic to $PSL(2, 9)$. More precisely, the map φ with

$$\varphi := \begin{cases} U_1 \to \bar{U} \\ U_2 \to \bar{\Omega} \\ V \to \bar{V} \\ W \to \bar{W} \end{cases} \tag{1}$$

extends to an isomorphism from $PSL(2, 9)$ into $PGL(3, q)$. Therefore, the group generated by $\bar{U}, \bar{\Omega}, \bar{V}$ and $\bar{W}$ is taken for Γ; that is,

$$\Gamma = \langle \bar{U}, \bar{\Omega}, \bar{V}, \bar{W} \rangle. \tag{2}$$

A representative system of the 90 cosets of $\langle W \rangle$ in $PSL(2,9)$ is listed below.

$$\begin{aligned}&\{I, VWUW, WUV\Omega VU, \Omega VWU, UWUV\Omega U^2, UWV\Omega U^2, V\Omega V\Omega, WV\Omega VU,\\ &W^2\Omega UV\Omega, VW^2\Omega U, \Omega VUV\Omega, W^2V\Omega UVU, V\Omega VW^2UW, \Omega VU^2VU,\\ &WV\Omega^2UV\Omega, VU, \Omega^2VU^2VU, V\Omega VU^2VU, V\Omega VW^2U, V\Omega U^2, U^2V\Omega^2VU^2,\\ &U^2, \Omega VW\Omega V\Omega, WVUV\Omega, V\Omega^2VU^2, W\Omega V\Omega, \Omega^2UVW^2U, WUWV\Omega UV,\\ &V\Omega^2VUV, UW^2V\Omega, \Omega VU, WUWUV\Omega, VU^2V\Omega UV, \Omega^2UVU, UWV\Omega^2VU^2,\\ &W^2UWVUV, VWV\Omega VU^2, \Omega^2VUV\Omega, WU^2V\Omega VU, \Omega^2U^2, VW^2\Omega V\Omega U,\\ &UVU^2V\Omega^2, UVW\Omega VU, W^2UV\Omega^2V, VWU^2V\Omega^2, \Omega UVW^2U, W^2UWV^2, V\Omega,\\ &\Omega UVW\Omega, W^2UVU^2, W^2V^2U, V\Omega VW\Omega, V\Omega^2U^2, VWV\Omega VUV, V\Omega UVW,\\ &UW^2V^2\Omega, VWV\Omega V, W^2VUV, UVW^2, UWV, UV, V, V\Omega UVW^2, W^2V^2\Omega,\\ &\Omega^2V\Omega, W^2V\Omega^2UV, WV\Omega VW, W^2V\Omega UV, W^2VU^2, V\Omega VW^2, W^2V\Omega^2U,\\ &UWV^2U, \Omega UVUWU^2, \Omega^2VU^2, \Omega VWV^2\Omega U, \Omega VW^2UW, V^2U^2V\Omega^2,\\ &UV\Omega VW^2, \Omega VWUV\Omega, \Omega VWV\Omega V, W\Omega UVU, \Omega VW\Omega VUV, U^2VWU,\\ &W^2UV\Omega, VWV\Omega U, UV\Omega V, W^2\Omega UV, V\Omega V\Omega U^2V, WUWUV, \Omega UVW^2\}.\end{aligned} \tag{3}$$

Replacing U, V, W, Ω with $\bar{U}, \bar{V}, \bar{W}.\bar{\Omega}$ gives a representative system of $\langle \bar{W} \rangle$ in Γ.

3 The fixed points of $\bar{W}$

The characteristic polynomial of $\mathbf{W}$ is $(\lambda^2-3)(\lambda-(1+2t))$ which has three pairwise distinct roots, as $p \neq 3$. Let s be an element in $GF(q)$ or in a quadratic extension $GF(q^2)$ such that $s^2 = 3$. Then

$$\mathbf{v}_1 = \big(1, \frac{1}{2}(s-1), \frac{1}{2}(s-1)\big), \quad \mathbf{v}_2 = \big(1, -\frac{1}{2}(s+1), -\frac{1}{2}(s+1)\big), \quad \mathbf{v}_3 = \big(0,1,-1\big)$$

are three independent eigenvectors of $\mathbf{W}$. For $i = 1,2,3$, let P_i be the point represented by $\mathbf{v}_i$. Then P_i are the fixed points of $\bar{W}$ in $PG(2,q)$ (or in $PG(2,q^2)$ when $s \in GF(q^2) \setminus GF(q)$). The subgroup S_2 of Γ generated by $\bar{V}$ and $\bar{W}$ is a dihedral group of order 8. Since $\bar{V}$ fixes P_3, this shows that S_2 is contained in the stabilizer of P_3 in the action of Γ. But this is not consistent with the hypothesis on the 1-point stabilizer in Theorem 1.1. Furthermore, $\bar{V}$ interchanges the points P_1 and P_2. Therefore, the Γ-orbit of P_1 contains P_2. From the classification of subgroups of A_6, every proper subgroup of Γ containing $\bar{W}$ also contains $\bar{V}$. From this, the stabilizer of P_1 under the action of Γ is the group of order 4 generated by $\bar{W}$. So, from now on we may limit ourselves to consider the Γ-orbit $\mathcal{O}$ of P_1. We stress that $\mathcal{O}$ is in $PG(2,q)$ (or in $PG(2,q^2)$ when $s \notin GF(q)$). The 90 points in $\mathcal{O}$ can be computed as the images of the $P_1 = (1, \frac{1}{2}(s-1), \frac{1}{2}(s-1))$ by the projectivities in the list in (3) after replacing U, V, W, Ω with $\bar{U}, \bar{V}, \bar{W}, \bar{\Omega}$. These points are listed below.

$(2,-s-1,-s-1)$; $((-12*s-12)*z*t+(-6*s-6)*z-6*s-18, (6*z+6*s)*t-6*z+6*s, (6*z-6*s)*t+12*z)$,
$(((6*s+18)*z+18*s+54)*t+(12*s+36)*z-36*s+36, ((6*s+18)*z+18*s-18)*t+(-6*s-18)*z+18*s+54, ((6*s-18)*z+18*s+18)*t+(-24*s-36)*z-36)$;
$((2*s+6)*z*t+(s+3)*z+9*s+3, (2*s+6)*z*t+(s+3)*z-9*s-15, (2*s-6)*z*t+(s-3)*z+3*s+3)$;
$(((s+3)*z+9*s+3)*t+(2*s+6)*z+6*s-6, ((-5*s-3)*z+3*s+9)*t+(-4*s-6)*z+6, ((-2*s-6)*z-12)*t+(-s-3)*z-3*s-9)$;
$((-2*s*z-6)*t+2*s*z+6*s+12, (-2*s-6)*z*t+(-s-3)*z+3*s-$

$3, (-2*s*z+6)*t-4*s*z+6*s+18);$
$((12*s+12)*z*t+(6*s+6)*z+6*s-18, (12*z-24*s-36)*t-12*z+12*s, (12*z+24*s+36)*t+24*z+36*s+36);$
$((-12*s*z+36)*t-24*s*z+36*s+108, (-12*s*z-36)*t+12*s*z+36*s+72, (-12*s-36)*z*t+(-6*s-18)*z+18*s-18);$
$(((3*s+15)*z+9*s+9)*t+(-3*s+3)*z+3*s+9, ((-3*s-3)*z+3*s-9)*t+(3*s+3)*z-9*s-9, ((6*s+6)*z+6*s-18)*t+(3*s+3)*z-3*s-27);$
$((6*z-6*s)*t+12*z, (6*z+6*s)*t-6*z+6*s, (-12*s-12)*z*t+(-6*s-6)*z-6*s-18);$
$(((6*s+6)*z+18*s+18)*t+(-6*s-6)*z-6*s+18, ((-6*s+6)*z-6*s-18)*t+(6*s+30)*z-18*s-18, ((6*s+6)*z+(6*s+54))*t+(12*s+12)*z-12*s+36);$
$(((18*s+54)*z+18*s-90)*t+36*s+36, ((-54*s-54)*z-18*s-90)*t-36*s*z-36*s-72, ((-18*s-54)*z+18*s-90)*t+(-18*s-54)*z+90*s-18);$
$((36*s+108)*z*t+(18*s+54)*z-162*s-270, ((-18*s+54)*z-54*s-54)*t+(18*s-54)*z-54*s-54, ((-18*s-54)*z+162*s+54)*t+(-36*s-108)*z);$
$(((6*s+6)*z+18*s+18)*t+(12*s+12)*z+12*s-36, ((-6*s-30)*z-18*s-18)*t+(6*s-6)*z-6*s-18, ((6*s+6)*z-18*s-18)*t+(12*s+12)*z-24*s);$
$(((18*s+18)*z-30*s+18)*t+-24*s-72, (-12*s-36)*t+(18*s+18)*z+30*s-18, ((-18*s-18)*z+6*s-18)*t+36*z-24*s-36);$
$(-4*z*t-2*z-2*s, -4*z*t-2*z-2*s, (-4*s-4)*z*t+(-2*s-2)*z-2*s-6);$
$(((6*s+6)*z-18*s-18)*t+(-6*s-6)*z-6*s-54, ((-6*s-30)*z+18*s+18)*t+(-12*s-24)*z+12*s, ((6*s+6)*z+18*s+18)*t+(-6*s-6)*z-6*s+18);$
$(((108*s+108)*z+36*s+180)*t+72*s*z+72*s+144, ((36*s+108)*z-36*s+180)*t+(36*s+108)*z-180*s+36, ((-36*s-108)*z-36*s+180)*t+-72*s-72);$
$((36*z+36*s)*t-36*z-72*s-108, (36*z-36*s)*t+72*z-108*s-108, (36*s+36)*z*t+(18*s+18)*z+18*s-54);$
$((2*z+2*s)*t-2*z+2*s, (-4*s-4)*z*t+(-2*s-2)*z-2*s-6, (2*z-2*s)*t+4*z);$
$(((-6*s-6)*z-18*s-18)*t+(-12*s-12)*z-12*s+36, ((6*s-6)*z-6*s-18)*t+(12*s+24)*z+12*s, ((-6*s-6)*z+6*s-18)*t+(6*s+6)*z-18*s-18); (-s-1, 2, -s-1);$
$(((-6*s+18)*z+18*s+18)*t+(-30*s-18)*z+18*s+54, ((-6*s-18)*z-18*s-54)*t+(6*s+18)*z-18*s+18, ((-6*s-18)*z-54*s-18)*t+(-12*s-36)*z-36*s+36);$
$((12*s+36)*z*t+(6*s+18)*z-54*s-90, ((-6*s+18)*z+18*s+18)*t+(-12*s+36)*z, ((-6*s-18)*z-54*s-18)*t+(6*s+18)*z-54*s-18);$
$((12*z-12*s)*t+24*z-36*s-36, (12*s+12)*z*t+(6*s+6)*z+6*s-18, (12*z+12*s)*t-12*z-24*s-36);$
$((-2*s-6)*z*t+(-s-3)*z+3*s-3, (-2*s*z-6*s-12)*t-4*s*z-6*s-18, (-2*s*z+6*s+12)*t+2*s*z-6);$
$(((-3*s-3)*z+9*s+9)*t+(3*s+3)*z+3*s+27, ((-3*s-3)*z-9*s-9)*t+(3*s+3)*z+3*s-9, ((3*s+15)*z-9*s-9)*t+(6*s+12)*z-6*s);$
$(((-18*s-54)*z+(18*s-90))*t+(-18*s-54)*z-18*s-126, ((18*s+54)*z+18*s-90)*t+-72*s-72, ((54*s+54)*z-18*s-90)*t+(18*s+54)*z+18*s-18);$
$(((-36*s-108)*z-36*s+180)*t+-72*s-72, ((36*s+108)*z-36*s+180)*t+(36*s+108)*z-180*s+36, ((108*s+108)*z+36*s+180)*t+72*s*z+72*s+144);$
$((-6*s+6)*z*t+(-3*s+3)*z+3*s+9, ((-3*s-3)*z+(3*s+27))*t+(-6*s-6)*z, ((-3*s-3)*z+(15*s+27))*t+(3*s+3)*z+15*s+27);$
$((-4*z-6*s-6)*t-2*z-2*s, (-4*z+(6*s+6))*t-2*z+4*s+6, (2*s+2)*z*t+(s+1)*z+s-3);$
$(((3*s+15)*z-9*s-9)*t+(6*s+12)*z-6*s, ((6*s+6)*z+12*s)*t+(3*s+3)*z+9*s+9, ((-3*s-3)*z-3*s-27)*t+(-6*s-6)*z+6*s-18);$
$(((36*s+108)*z+108*s+36)*t+216*s+72, (-72*s-72)*t+(36*s-108)*z-36*s-36, ((-36*s-108)*z-108*s-180)*t+(-36*s-108)*z+108*s+180);$
$(((-s-1)*z-3*s-3)*t+(s+1)*z+s-3, ((-s-1)*z+(3*s+3))*t+(s+1)*z+s+9, ((s+5)*z-3*s-3)*t+(2*s+4)*z-2*s);$
$((-12*s*z-36*s-72)*t-24*s*z-36*s-108, (-12*s-36)*z*t+(-6*$

$s-18)*z+18*s-18,(-12*s*z+(36*s+72))*t+12*s*z-36);$
$(((-18*s-54)*z+54*s+162)*t+(-36*s-108)*z+216,((90*s+54)*z+$
$54*s+162)*t+(18*s-54)*z+54*s+54,((-18*s-54)*z-54*s-162)*t+$
$(-36*s-108)*z+108*s-108);$
$((-144*s*z+216*s+648)*t-72*s*z+216*s+432,(72*s+216)*z*t+(36*$
$s+108)*z-108*s+108,(-144*s*z-216*s-648)*t-72*s*z-216);$
$(((6*s+6)*z-18*s-18)*t+(12*s+12)*z-24*s,((12*s+24)*z+12*s)*t+(6*$
$s+30)*z+18*s+18,((-12*s-12)*z-12*s+36)*t+(-6*s-6)*z+6*s+54);$
$(((6*s+18)*z-18*s+18)*t+(12*s+36)*z+72,((6*s+18)*z-18*s-54)*t+$
$(-6*s-18)*z-54*s-18,((6*s-18)*z-18*s-18)*t+(30*s+18)*z-18*s-54);$
$((-s-1)*t,2,(s+1)*t+s+1);((36*z-72*s-108)*t-36*z+36*s,(36*z+(72*$
$s+108))*t+72*z+108*s+108,(36*s+36)*z*t+(18*s+18)*z+18*s-54);$
$((-12*s-12)*z*t+(-6*s-6)*z-6*s-54,((6*s+6)*z-30*s-54)*t+$
$(-6*s-6)*z-30*s-54,((-6*s+6)*z-6*s-18)*t+(-12*s+12)*z);$
$((12*s*z+(36*s+72))*t+24*s*z+36*s+108,(12*s*z-36*s-72)*t-$
$12*s*z+36,(12*s+36)*z*t+(6*s+18)*z-18*s+18);$
$(((18*s-18)*z+18*s+54)*t+(-18*s-90)*z+54*s+54,((-18*s-18)*z+(54*s+54))*$
$t+(18*s+18)*z+18*s+162,((-18*s-18)*z-18*s+54)*t+(-36*s-36)*z-72*s);$
$((-12*s-36)*z*t+(-6*s-18)*z+54*s+90,((6*s+18)*z+(54*s+18))*$
$t+(-6*s-18)*z+54*s+18,((6*s-18)*z-18*s-18)*t+(12*s-36)*z);$
$(((-3*s-3)*z-9*s-9)*t+(-6*s-6)*z-6*s+18,((-3*s-3)*z+(9*s+9))*$
$t+(-6*s-6)*z+12*s,((3*s+15)*z+(9*s+9))*t+(-3*s+3)*z+3*s+9);$
$(-216*s,(-108*s-324)*t+-108*s-324,(108*s+324)*t);$
$((-4*s-4)*z*t+(-2*s-2)*z-2*s-6,(2*z-2*s)*t+4*z,(2*z+2*s)*t-2*z+2*s);$
$(((-s+3)*z+3*s+3)*t+(-5*s-3)*z+3*s+9,((-s-3)*z-9*s-3)*$
$t+(-2*s-6)*z-6*s+6,((-s-3)*z-3*s-9)*t+(s+3)*z-3*s+3);$
$((6*s+6)*z*t+(3*s+3)*z+3*s+27,(-6*s+6)*z*t+(-3*s+3)*z+3*$
$s+9,(6*s+6)*z*t+(3*s+3)*z-15*s-27);$
$(-108*s-108,-108*s-108,216);$
$((12*s+36)*z*t+(6*s+18)*z-54*s-90,((-6*s-18)*z+54*s+18)*t+$
$(-12*s-36)*z,((-6*s+18)*z-18*s-18)*t+(6*s-18)*z-18*s-18);$
$((2*z-2*s)*t+4*z,(-4*s-4)*z*t+(-2*s-2)*z-2*s-6,(2*z+2*s)*t-2*z+2*s);$
$(((216*s+648)*z+(648*s+1944))*t+(-216*s-648)*z+648*s-648,((-1080*$
$s-648)*z+(648*s+1944))*t+(-864*s-1296)*z+1296,((216*s+648)*z-$
$648*s-1944)*t+(-216*s-648)*z-1944*s-648);$
$(((-18*s-18)*z+6*s-90)*t+(-18*s-18)*z+30*s-18,(-36*z+(24*$
$s+36))*t+(-18*s-54)*z+30*s+18,(-12*s-36)*t+(18*s+18)*z+30*$
$s-18);(-108*s-108,216*t,(108*s+108)*t+108*s+108);$
$((72*s+216)*z*t+(36*s+108)*z-108*s+108,(-144*s*z-216*s-648)*$
$t-72*s*z-216,(-144*s*z+(216*s+648))*t-72*s*z+216*s+432);$
$((-36*s-36)*z*t+(-18*s-18)*z+90*s+162,(36*s-36)*z*t+(18*s-18)*z-18*$
$s-54,(-36*s-36)*z*t+(-18*s-18)*z-18*s-162);((-6*s+6)*z*t+(-3*s+3)*$
$z+3*s+9,(6*s+6)*z*t+(3*s+3)*z+3*s+27,(6*s+6)*z*t+(3*s+3)*z-15*s-27);$
$((-2*s-6)*z*t+(-s-3)*z+3*s-3,(4*s*z-6*s-18)*t+2*s*z-6*$
$s-12,(4*s*z+(6*s+18))*t+2*s*z+6);$
$((-2*s+2)*z*t+(-s+1)*z+s+3,(2*s+2)*z*t+(s+1)*z-5*s-9,(2*$
$s+2)*z*t+(s+1)*z+s+9);$
$((-4*s-4)*z*t+(-2*s-2)*z-2*s-6,-4*z*t-2*z-2*s,-4*z*t-2*z-2*s);$
$(((-18*s-54)*z+(18*s-90))*t+(-18*s-54)*z+90*s-18,((-54*s-$
$54)*z-18*s-90)*t-36*s*z-36*s-72,((18*s+54)*z+(18*s-90))*t+$
$36*s+36);(216,(-108*s-108)*t,(108*s+108)*t+108*s+108);$
$((2*s+2)*z*t+(s+1)*z+s-3,(2*z-2*s)*t+4*z-6*s-6,(2*z+2*s)*t-2*z-4*s-6);$
$(((-18*s-54)*z-18*s+90)*t+72*s+72,((18*s+54)*z-18*s+90)*t+(18*s+$
$54)*z+18*s+126,((-54*s-54)*z+(18*s+90))*t+(-18*s-54)*z-18*s+18);$

$((-36*s-36)*z*t+(-18*s-18)*z+90*s+162, (-36*s-36)*z*t+(-18*s-18)*z-18*s-162, (36*s-36)*z*t+(18*s-18)*z-18*s-54);$
$(((-18*s-54)*z+(18*s-90))*t+(-18*s-54)*z+90*s-18, ((18*s+54)*z+(18*s-90))*t+36*s+36, ((-54*s-54)*z-18*s-90)*t-36*s*z-36*s-72);$
$(-12*z*t-6*z-6*s, (-12*s-12)*z*t+(-6*s-6)*z-6*s-18, -12*z*t-6*z-6*s);$
$((36*s+36)*z*t+(18*s+18)*z+18*s-54, (36*z+36*s)*t-36*z-72*s-108, (36*z-36*s)*t+72*z-108*s-108);$
$((6*z+6*s)*t-6*z+6*s, (6*z-6*s)*t+12*z, (-12*s-12)*z*t+(-6*s-6)*z-6*s-18);$
$((-36*s-108)*t+-36*s-108, (36*s+108)*t, -72*s);$
$(((-s-3)*z+3*s-3)*t+(-2*s-6)*z-12, ((-s+3)*z+(3*s+3))*t+(-5*s-3)*z+3*s+9, ((-s-3)*z+(3*s+9))*t+(s+3)*z+9*s+3);$
$((-4*z-6*s-6)*t-2*z-2*s, (2*s+2)*z*t+(s+1)*z+s-3, (-4*z+(6*s+6))*t-2*z+4*s+6);$
$(((-36*s-108)*z+(324*s+108))*t+(-72*s-216)*z, ((-36*s-108)*z+(324*s+540))*t+(36*s+108)*z+324*s+540, (72*s-216)*z*t+(36*s-108)*z+108*s+108);$
$((6*s-18)*z*t+(3*s-9)*z+9*s+9, ((-3*s-9)*z+(27*s+45))*t+(3*s+9)*z+27*s+45, ((-3*s-9)*z+(27*s+9))*t+(-6*s-18)*z);$
$((-72*s+72)*z*t+(-36*s+36)*z+36*s+108, ((-36*s-36)*z-36*s-324)*t+(36*s+36)*z-36*s-324, ((-36*s-36)*z-180*s-324)*t+(-72*s-72)*z);$
$(((18*s-18)*z-18*s-54)*t+(36*s+72)*z+36*s, ((-18*s-18)*z+(18*s-54))*t+(18*s+18)*z-54*s-54, ((-18*s-18)*z-54*s-54)*t+(-36*s-36)*z-36*s+108);$
$((-12*s-36)*z*t+(-6*s-18)*z-54*s-18, ((6*s-18)*z-18*s-18)*t+(12*s-36)*z, ((6*s+18)*z-54*s-90)*t+(-6*s-18)*z-54*s-90);$
$(((-36*s+108)*z+(108*s+108))*t+(-180*s-108)*z+108*s+324, ((72*s+216)*z+(216*s-216))*t+(36*s+108)*z-108*s-324, ((72*s+216)*z+432)*t+(36*s+108)*z+108*s+324);$
$(((s+3)*z-3*s-9)*t+(2*s+6)*z-12, ((s+3)*z+(3*s+9))*t+(2*s+6)*z-6*s+6, ((-5*s-3)*z-3*s-9)*t+(-s+3)*z-3*s-3);$
$(((108*s+108)*z-180*s+108)*t+72*s+216, (144*s+432)*t+(108*s+108)*z+180*s-108, (216*z+(144*s+216))*t+(-108*s-108)*z-36*s+108);$
$((-4*s*z+(6*s+18))*t-2*s*z+6*s+12, (-4*s*z-6*s-18)*t-2*s*z-6, (2*s+6)*z*t+(s+3)*z-3*s+3);$
$((-6*s+6)*z*t+(-3*s+3)*z+3*s+9, ((-3*s-3)*z-15*s-27)*t+(-6*s-6)*z, ((-3*s-3)*z-3*s-27)*t+(3*s+3)*z-3*s-27);$
$((-36*s-108)*t, (36*s+108)*t+36*s+108, 72*s);$
$(((6*s-6)*z-6*s-18)*t+(12*s+24)*z+12*s, ((-6*s-6)*z-18*s-18)*t+(-12*s-12)*z-12*s+36, ((-6*s-6)*z+(6*s-18))*t+(6*s+6)*z-18*s-18);$
$(((3*s+15)*z+9*s+9)*t+(-3*s+3)*z+3*s+9, ((-3*s-3)*z+(9*s+9))*t+(-6*s-6)*z+12*s, ((-3*s-3)*z-9*s-9)*t+(-6*s-6)*z-6*s+18);$
$((72*s*z-72*s-288)*t+72*s*z-144*s-360, (72*s-72)*t+(36*s+108)*z+36*s-36, (-72*s*z-72*s-288)*t+72*s+72);$
$(((3*s+15)*z-9*s-9)*t+(6*s+12)*z-6*s, ((-3*s-3)*z-9*s-9)*t+(3*s+3)*z+3*s-9, ((-3*s-3)*z+(9*s+9))*t+(3*s+3)*z+3*s+27);$
$(((3*s+15)*z+9*s+9)*t+(-3*s+3)*z+3*s+9, ((-3*s-3)*z-9*s-9)*t+(-6*s-6)*z-6*s+18, ((-3*s-3)*z+(9*s+9))*t+(-6*s-6)*z+12*s)).$

Let $\mathcal{O} = \{P_1, Q_1, \ldots, Q_{89}\}$. The points P_1, Q_i and Q_j are collinear if and only if the determinant $D_{i,j}$ of the coordinates of these points vanishes. There are 3916 triples $\{P_1, Q_i, Q_j\}$ with $1 \leq i < j \leq 89$. Observe that $D_{i,j}$ can be viewed as a polynomial in t, s and z, say $D_{i,j}(t, s, z)$, with coefficients in $\mathbb{Z}$. Therefore a necessary and sufficient condition for the points $Q_i, Q_j \in \mathcal{O}$ to produce together with P_1 a collinear triple is that (t, s, z) be a solution of the

system of equations

$$\begin{cases} t^2+t+1=0; \\ s^2=3; \\ z^2=5; \\ D_{i,j}(t,s,z)=0. \end{cases} \tag{4}$$

We look at the above system over $\mathbb{Z}$ with unknowns t, s, z and use Sylvester's resultant to discuss solvability. Eliminating t from the first and the forth equations produces an equation in s, z over $\mathbb{Z}$: then eliminating s from this and the second equation provides an equation in z over $\mathbb{Z}$; finally eliminating z from this and the third equation gives an integer, the resultant of the system. A sufficient condition for a triple of points not to be collinear is that this resultant does not vanish in Z_p.
A computer aided search shows that such a resultant is a non zero integer for any of the above 3916 cases. Now, let $\delta_{i,j}$ be the set of all prime divisors of the resultant arising from the triple $\{P_1, Q_i, Q_j\}$. If $p \notin \delta_{i,j}$ then the points P_1, Q_i, Q_j are not collinear. More generally, let δ be the set of all primes appearing in some of the 3916 sets $\delta_{i,j}$. If $p \notin \delta$ then the Γ-orbit $\mathcal{O}$ is 90-arc.

An exhaustive computer-aided computation shows that δ has size 14, namely $\delta = \{2, 3, 5, 7, 11, 13, 17, 19, 61, 109, 181, 229, 241, 421\}$. Therefore, the following result holds.

Proposition 3.1. *The Γ-orbit $\mathcal{O}$ of the point P_1 has length 90 and the stabilizer of P_1 in Γ is a cyclic group of order 4. Furthermore, $\mathcal{O}$ is a 90-arc on $PG(2, q)$ with $q = p^h$ and $p \geq 7$ apart from finitely many values of p which are*

$$7, 11, 13, 17, 19, 61, 109, 181, 229, 241, 421.$$

Now, we discuss the exceptional cases.

3.1 $p = 7, 11, 13, 17$

In this case $p^2 \equiv 1$ or $19 \pmod{30}$. Therefore, $\mathcal{O}$ lies in $PG(2, p^2) \setminus PG(2, p)$. By a MAGMA [3] computation, some $\delta_{i,j}$ is divisible by p. Hence $\mathcal{O}$ is not an arc. Some more effort allows to compute the intersection numbers of $\mathcal{O}$ with lines. The results are reported below.

- For $q = 7^2$ a square root of 5 is w^{20}, where w is a primitive element of $GF(7^2)$ such that $w^2+6w+3 = 0$. In this case $\mathcal{O}$ is a complete $(90, 4)-$arc with 336 external lines, 810 tangents, 765 bi-secants, 540 four-secants.

- For $q = 11^2$ a primitive cubic root of unity is w^{40}, where w is a primitive element of $GF(11^2)$. In this case $\mathcal{O}$ is a non-complete $(90, 5)-$arc with 7248 external lines, 4320 tangents, 3105 bi-secants, 90 five-secants.

- For $q = 13^2$ a square root of 5 is w^{63}, where w is a primitive element of $GF(13^2)$ such that $w^2 + 12w + 2 = 0$. In this case $\mathcal{O}$ is a non-complete $(90, 4)-$arc with 16896 external lines, 8730 tangents, 2925 bi-secants, 180 four-secants.

- For $q = 17^2$ a primitive cubic root of unity is w^{98} and a square root of 5 is w^{45}, where w is a primitive element of $GF(17^2)$ such that $w^2+16w+3 = 0$. In this case $\mathcal{O}$ is a non-complete $(90, 3)-$arc with 61356 external lines, 19170 tangents, 2925 bi-secants, 360 three-secants.

3.2 $p = 19$

Γ is a projectivity group of $PG(2,19)$. However, $s \in GF(19^2) \setminus GF(19)$, whence $\mathcal{O}$ lies in $PG(2,19^2) \setminus PG(2,19)$. Furthermore, $\mathcal{O}$ is a non-complete $(90,5)-$arc with 101676 external lines, 25650 tangents, 3285 bi-secants, 72 five-secants. Here, $s = w^{130}$ where $w^2 + 18w + 2 = 0$.

3.3 $p = 61, 109, 181, 229, 241, 421$

In this case, Γ is a projective group of $PG(2,p)$ and $s \in GF(p)$. Therefore $\mathcal{O}$ lies in $PG(2,p)$. Again, some $\delta_{i,j}$ is divisible by p, and $\mathcal{O}$ is not an arc. By a MAGMA computation, the intersection numbers of $\mathcal{O}$ lines can be computed, and the results are reported below.

- For $p = 61$ $\mathcal{O}$ is a non-complete $(90,6)-$arc with 1068 external lines, 450 tangents, 2025 bi-secants, 180 four-secants, 60 six-secants.
- For $p = 109$ $\mathcal{O}$ is a non-complete $(90,3)-$arc with 5736 external lines, 2970 tangents, 2925 bi-secants, 360 three-secants.
- For $p = 181$ $\mathcal{O}$ is a non-complete $(90,3)-$arc with 20208 external lines, 9450 tangents, 2925 bi-secants, 360 three-secants.
- For $p = 229$ $\mathcal{O}$ is a non-complete $(90,4)-$arc with 35436 external lines, 14130 tangents, 2925 bi-secants, 180 four-secants.
- For $p = 241$ $\mathcal{O}$ is a non-complete $(90,4)-$arc with 40008 external lines, 15210 tangents, 2925 bi-secants, 180 four-secants.
- For $p = 421$ $\mathcal{O}$ is a non-complete $(90,3)-$arc with 143328 external lines, 31050 tangents, 2925 bi-secants, 360 three-secants.

The results of the present section provide a proof of Theorem 1.1.

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